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Generating Fuzzy Measures*

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1. INTRODUCTION

Fuzzy measure and integration theory was introduced by Sugeno [13] in order to deal with events affected by nonstatistical uncertainty. Further developments were concerned both with the analytical treatment of the concept of fuzzy measure—and its links with the classical measure theory (see, e.g., [3–5, 8, 9, 11, 14])—and the determination of a unique framework into which fuzziness and randomness can be set (see [7]). The main feature of the fuzzy measure in comparison with the classical one is the loss of additivity conditions, while continuity and monotonicity axioms are assumed. Our present aim is to exhibit some constructive examples of fuzzy measures, in terms of compositions of σ-additive measures and suitable real valued functions. Furthermore we show some properties that are peculiar to the related class of the fuzzy measures above.

2. CLOSURE PROPERTIES

We shall consider fuzzy measures in the sense of [13, 7], i.e., we shall adopt the following definition. Let X be a set and $\underline{A}$ be a σ-algebra of subsets of X. A *fuzzy measure* μ is defined as a nonnegative, extended real valued set function μ: $A \to [0, +\infty]$, with the properties:

F1. $\mu(\varnothing) = 0$

F2. $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$

F3. $A_1 \subset A_2 \subset \cdots,\ A_n \in \underline{A} \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \lim_n \mu(A_n)$ (lower semi-continuity)

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F4. $A_1 \supset A_2 \supset \cdots$, $A_n \in \underline{A}$, $\mu(A_1) < \infty \Rightarrow \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_n \mu(A_n)$ (upper semicontinuity).

The triple $(X, \underline{A}, \mu)$ is called a *fuzzy measure space.* For the measure theoretical notations and the recalled results see [2]. The following closure properties hold.

PROPOSITION 1.1. *Let $(X, \underline{A})$ be a measurable space, $\mu_1, ..., \mu_n$ a finite sequence of fuzzy measures on $(X, \underline{A})$ and $c_1, ..., c_n$ nonnegative real numbers. Then the set functions $\bar{v} = \bigvee \mu_i$ and $\mu = \sum c_i \mu_i$ are fuzzy measures. Furthermore, if all measures μ_i's are finite also $\bar{\bar{v}} = \bigwedge \mu_i$ is a fuzzy measure.*

Proof. Let us prove just that $\bar{v}$ is a fuzzy measure. To ease the notations let us consider, without loss of generality, the case $n = 2$. It is $\bar{v}(\emptyset) = \mu_1(\emptyset) \vee \mu_2(\emptyset) = 0$.

For every $A, B \in \underline{A}$, $A \subset B$ implies

$$\bar{v}(A) = \mu_1(A) \vee \mu_2(A) \leqslant \mu_1(B) \vee \mu_2(B) = \bar{v}(B).$$

In order to prove F3 let (A_k) be an increasing sequence of $\underline{A}$:

$$\bar{v}\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu_1\left(\bigcup_{k=1}^{\infty} A_k\right) \vee \mu_2\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_k \mu_1(A_k) \vee \lim_k \mu_2(A_k)$$
$$= \bigvee_{k=1}^{\infty} (\mu_1(A_k) \vee \mu_2(A_k)) = \lim_k \bar{v}(A_k).$$

The upper semicontinuity can be shown as follows. Let (A_k) be a decreasing sequence in $\underline{A}$, $\bar{v}(A_1) < \infty$. It is

$$\bar{v}\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_k \mu_1(A_k) \vee \lim_k \mu_2(A_k) = \bigwedge_{k=1}^{\infty} (\mu_1(A_k) \vee \mu_2(A_k))$$
$$= \lim_k \bar{v}(A_k),$$

as the sequence $(\mu_i(A_k))_k$ is decreasing, for every i.

The following Nikodým-like property [6] holds.

PROPOSITION 1.2. *Let $(X, \underline{A})$ be a measurable space and (μ_n) a sequence of fuzzy measures which converges uniformly with respect to $A \in \underline{A}$ to a set function μ. Then μ is a fuzzy measure.*

Proof. Conditions (F1) and (F2) are evidently satisfied. From the equalities

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_n \mu_n\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_n \lim_k \mu_n(A_k) = \lim_k \lim_n \mu_n(A_k),$$

—the last equality being guaranteed by uniform convergence of the μ_n's—upper semicontinuity follows. Analogously lower semicontinuity can be shown.

3. Generating Fuzzy Measures by Compositions

Proposition 3.1. *Let μ be a fuzzy measure on the measurable space $(X, \underline{A})$, $g : [0, \mu(X)] \to R^+$ an increasing nonnegative continuous function, with $g(0) = 0$. Then the set function $g \circ \mu$ is a fuzzy measure on $(X, \underline{A})$.*

Proof. Conditions (F1) and (F2) trivially hold. Furthermore, for every increasing sequence (A_n) in $\underline{A}$,

$$g \circ \mu \left(\bigcup A_n \right) = g\,(\lim_n \mu(A_n)) = \lim_n g(\mu(A_n)).$$

Analogously the upper semicontinuity can be shown.

Corollary 3.1. *Let μ be a σ-additive measure on the measurable space $(X, \underline{A})$ and $g : [0, \mu(X)] \to R^+$ an increasing nonnegative continuous function, with $g(0) = 0$. Then $g \circ \mu$ is a fuzzy measure on $(X, \underline{A})$.*

Remark 3.1. The corollary above allows us to construct examples of fuzzy measures from well-known real functions and measures, e.g., e^{μ}, $\arctan \mu$, $\log(1 + \mu)$, μ^a, for $a > 0$ and

$$g(\mu(A)) = \int_0^{\mu(A)} f(x)\, d\mu',$$

where μ' is a Lebesgue measure in R, and $f(x) \geqslant 0$ integrable in $[0, \mu(X)]$.

Let us remark that the supremum of a family of fuzzy measures is not, in general, a fuzzy measure for the upper semicontinuity can fail [9].

In order to show a sufficient condition for a supremum of fuzzy measures to be itself a fuzzy measure, let us recall the following definition [1].

Let $(X, \underline{A})$ be a measurable space and $(\mu_\alpha)_\alpha$ an arbitrary family of σ-additive measures on $(X, \underline{A})$. The family $(\mu_\alpha)_\alpha$ is said to be *uniformly additive* if for every sequence $(X_n) \subset \underline{A}$ such that $\lim X_n = \varnothing$, then

$$\lim_n \sup_\alpha \mu_\alpha(X_n) = 0.$$

Proposition 3.2. *Let $(X, \underline{A})$ be a measurable space and (μ_α) a family of uniformly additive measures such that* $\sup(\mu_\alpha) < +\infty$. *Then the set function*

$$\bar{\mu} : A \in \underline{A} \to \sup_\alpha \mu_\alpha(A)$$

is a sub-additive fuzzy measure on $(X, \underline{A})$.

Proof. Properties (F1), (F2), (F3), and sub-additivity of $\bar{\mu}$ follow easily (see also [9]).

Let now prove the above semicontinuity of $\bar{\mu}$. If (X_n) is a decreasing sequence in $\underline{A}$ such that $X_n \downarrow X$, setting $Y_n = X_n - X$, then, by sub-additivity of $\bar{\mu}$ and uniform additivity of (μ_α), we have

$$\lim \bar{\mu}(X_n) \leqslant \lim \bar{\mu}(Y_n) + \bar{\mu}(X) = \bar{\mu}(X).$$

On the other hand, by monotonicity of $\bar{\mu}$,

$$\lim \bar{\mu}(X_n) \geqslant \bar{\mu}(X).$$

We are now able to prove the announced result.

PROPOSITION 3.3. *Let $(X, \underline{A})$ be a measurable space, (μ_α) a uniform additive family on $(X, \underline{A})$ such that* $\sup_\alpha \mu_\alpha(X) < +\infty$ *and $g : R^+ \to R^+$ an increasing continuous function with* $g(0) = 0$. *Then the set function*

$$v : A \in \underline{A} \to \sup_\alpha g(\mu_\alpha(A))$$

is a fuzzy measure.

Proof. By Proposition 3.2 $\sup \mu_\alpha$ is a fuzzy sub-additive measure. Then, by continuity of g and Proposition 3.1, v is a fuzzy measure, too.

PROPOSITION 3.4. *Let $(X, \underline{A})$ be a measurable space (μ_n) a sequence of σ-additive finite measures on $(X, \underline{A})$ such that* $\lim \mu_n(A)$ *exists for every $A \in \underline{A}$ and $g : R^+ \to R^+$ an increasing continuous function with* $g(0) = 0$. *Then the function*

$$v : A \in \underline{A} \to \lim_n g(\mu_n(A))$$

is a fuzzy measure on $(X, \underline{A})$.

Proof. By Corollary 3.1 the $g(\mu_n)$'s are fuzzy measures on $(X, \underline{A})$. Furthermore, by Nikodym's theorem [6] the function $\mu = \lim \mu_n$ is a σ-additive measure. Then $g(\mu)$ is a fuzzy measure. Therefore by the continuity of g,

$$v = \lim_n g(\mu_n) = g(\mu).$$

Let us now mention some well-known properties of measure theory that remain valid in the fuzzy case.

PROPOSITION 3.5. *Let $(X, \underline{A})$ be a measurable space, μ a finite measure on $(X, \underline{A})$, g: $R^+ \to R^+$ an increasing continuous function with* $g(0) = 0$, *and*

(A_j), $j \in I$, a family of arbitrary cardinality (card J) *in $\underset{\sim}{A}$, such that $A_i \cap A_j = \emptyset$, $i \neq j$. Then*

$$\operatorname{card}\{i \in I : g(\mu(A_i)) > 0\} \leqslant \operatorname{card} N.$$

Proof. We just observe that

$$\{i \in I : g(\mu(A_i)) > 0\} \subset \{i \in I : \mu(A_i) > 0\},$$

and the last set has cardinality not greater than card N, as μ is a σ-additive measure.

Let $\lambda \in (-1, \infty)$ be a real number. A fuzzy measure μ on $(X, \underset{\sim}{A})$ is called λ-additive [13] if $\mu(X) = 1$, and, whenever $A, B \in \underset{\sim}{A}$ and $A \cap B = \emptyset$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\,\mu(B).$$

Remark 3.2. The validity of Proposition 3.5 can be proved also for super-additive fuzzy measures, for instance, when μ is λ-additive, with $\lambda > 0$.

Let us recall now the following result due to Kruse [4]. If μ is a λ-additive fuzzy measure, $\lambda \neq 0$, on $(X, \underset{\sim}{A})$, then

$$\mu^* = (\log(1 + \lambda\mu)) \cdot (\log(1 + \lambda))^{-1}$$

is a probability measure on $(X, \underset{\sim}{A})$.

The result allows us to prove the following proposition that, in some sense, inverts Corollary 3.1.

PROPOSITION 3.6. *Let $(X, \underset{\sim}{A})$ be a measurable space, μ a λ-additive fuzzy measure, $\lambda \neq 0$, then $\mu = g(\mu^*)$, where g satisfies the conditions of Proposition* 3.3., *and μ^* is a probability measure.*

Proof. It is sufficient to assume

$$g(y) = ((1 + \lambda)^y - 1)/\lambda$$

and recall the mentioned Kruse's result.

4. A Possible Extension

The concept of fuzzy measure can be enlarged by considering relative fuzzy measures.

DEFINITION 4.1. Let $(X, \underset{\sim}{A})$ be a measurable space. A real valued set function v defined in $\underset{\sim}{A}$ is said to be a relative fuzzy measure if $v(\emptyset) = 0$ and v is upper and lower semicontinuous.

A class of relative fuzzy measures can be determined following the composition procedure used in Section 3. Precisely it is easy to see that

PROPOSITION 4.1. *Let $(X, \underline{A})$ be a measurable space, μ a relative measure on $(X, \underline{A})$, $g : R \to R$ an increasing continuous function, with $g(0) = 0$. Then the set function $g \circ \mu$ is a relative fuzzy measure on $(X, \underline{A})$.*

A Hahn decomposition property holds. Indeed, under the previous hypotheses and the additional condition

$$yg(y) \geqslant 0,$$

it is $X = P \cup N$, where P and N are positive and negative sets, respectively, i.e., $g(\mu(A)) \geqslant 0$, for every $A \in \underline{A}$, $A \subset P$, and $g(\mu(A)) \leqslant 0$ for every $A \in \underline{A}$, $A \subset N$.

Note added in proof. Fuzzy measure and integration theory finds generalizations and improvements into the framework of the measures that are decomposable w.r. to triangular conorms (see S. WEBER, Decomposable measures and integrals for archimedean t-conorms, *J. Math. Anal. Appl.* **101**(1) (1984), 114–138; M. SQUILLANTE AND A. G. S. VENTRE, A Yoshida–Hewitt like theorem for decomposable measures, *Ricerche di Matematica* **37**(2) (1988), 203–212).

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